UNIVERSITY OF
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## SCHOOL OF MATHEMATICS AND PHYSICS

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# Mathematics Challenge–2021–22

## Brief solutions

*Note that each problem may have several different solutions by various methods.*

**Problem 1.** Suppose that in the year 2021 fossil fuel power generation, nuclear power stations, and renewable sources contributed to overall electricity generation in the ratio of 11:4:10, respectively. If in the year 2022 the fossil fuel power generation decreases by 5%, and nuclear increases by 2%, by how much must the renewable power generation increase in percentage in order to maintain the same level of overall electricity generation?

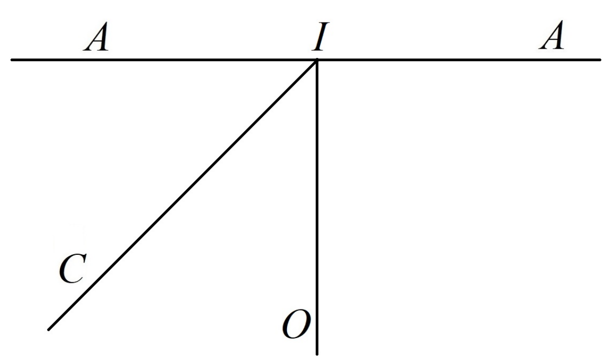
**Solution of Problem 1.** If $M$ is the total generation, we solve the equation

$$0.95 \cdot M\frac{11}{25} + 1.02 \cdot M\frac{4}{25} + \left(1 + \frac{x}{100}\right) \cdot M\frac{10}{25} = M,$$

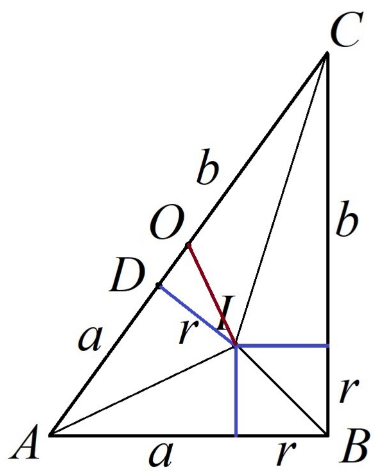
whence $x = 0.047 = 4.7\%$.

**Problem 2.** In a triangle $ABC$, let $I$ denote the centre of the inscribed circle, and $O$ the centre of the circumscribed circle. Given that $\angle AIO = 90°$ and $\angle CIO = 45°$, find the ratios of the lengths of the sides $AB : BC : CA$.

**Solution of Problem 2.** The picture shows two possible disposition of the rays $IA$, $IO$, $IC$:



It follows that $\angle AIC = 45°$ or $135°$. Since $AI$ and $CI$ are bisectors of $\angle BAC$ and $\angle BCA$, we have $\angle ABC = 180° - 2(180° - \angle AIC) = 2\angle AIC - 180°$. We cannot have $\angle AIC = 45°$, as then $\angle ABC = -90°$, impossible. Therefore $\angle AIC = 135°$, whence $\angle ABC = 90°$. Thus, this is a right triangle, and by the well-known property the centre $O$ of the circumscribed circle is the midpoint of $AC$. The perpendiculars dropped from $I$ onto the sides give rise to segments of lengths $a, b, r$ as shown on the picture (where $r$ is the radius of the inscribed circle):




By Pythagoras we have $(a+r)^2 + (b+r)^2 = (a+b)^2$, whence $2r^2 = 2ab - 2ar - 2br$. By similar triangles $\triangle AID$ and $\triangle AOI$ we have $\frac{\sqrt{a^2+r^2}}{a} = \frac{(a+b)/2}{\sqrt{a^2+r^2}}$, whence $2r^2 = ab - a^2$. Therefore, $ab - a^2 = 2ab - 2ar - 2br$, whence $r = a/2$. We also have $b = (2r^2 + a^2)/a = 3a/2$ after substituting $r = a/2$. We have $AB = a + r = 3a/2$ and $CB = b + r = 3a/2 + a/2 = 2a$. As a result $AB : CB = 3 : 4$, and another application of Pythagoras gives $AB : CB : AC = 3 : 4 : 5$.

**Comments on submissions and solutions of Problem 2.** It was important to deduce that the rays $IA$, $IO$, $IC$ are situated as we showed them to be; in some submissions this was assumed without due proof. Once the triangle is shown to be a right one, various ways of calculations are possible and indeed submitted.

**Problem 3.** If the sum of digits of a positive integer $A$ is 2021, and the sum of digits of a positive integer $B$ is 2022, what is the minimum possible sum of digits of the number $A+B$?

**Solution of Problem 3.** It is known that the remainder of a positive integer after division by 9 is the same as that of its sum of digits (follows from the fact that $10^k = \underbrace{99\ldots9}_{k} + 1$ for every positive integer $k$). Hence the remainder of $A+B$ after division by 9 is the same as of $2+0+2+1+2+0+2+2 = 11$, which is 2. Therefore the sum of digits of $A+B$ cannot be less than 2. And it is indeed possible to have this sum to be equal to 2, for example, for $A = \underbrace{4\,5\,4\,5\,4\,5\,\ldots\,4\,5\,4\,5}_{\text{45 repeated 224 times}}\,5$ and $B = 1\,\underbrace{5\,4\,5\,4\,5\,4\,\ldots\,5\,4\,5\,4}_{\text{54 repeated 224 times}}\,5$.

**Comments on submissions and solutions of Problem 3.** Some submissions only produced examples with $A+B$ having sum of digits 2, without proving that it cannot be less than 2. There are, of course, many possible examples with $A+B$ having sum of digits 2.

**Problem 4.** A small grasshopper starts jumping from a point $O$ on the plane. It makes its first jump of length 1 cm (along some straight line), then the second jump of length 2 cm in either of the two directions perpendicular to the first jump, then the third jump of 3 cm in either of the two directions perpendicular to the second, and so on, each time increasing the length of the jump by 1 cm and choosing one of the two directions perpendicular to the preceding jump.

(a) Can the grasshopper return to the origin $O$ after 71 jumps?
(b) Can the grasshopper return to the origin $O$ after 2022 jumps?

**Solution of Problem 4.** We assume without loss of generality that the first jump starts at the origin of a Cartesian plane in the positive direction of the $x$-axis. All jumps with odd numbers are represented by odd numbers $1, \pm 3, \pm 5, \dots$ and all jumps with even numbers are represented by even numbers $\pm 2, \pm 4, \dots$. The grasshopper returns to the origin exactly if both the sum of those positive or negative odd numbers is 0, and the sum of those positive or negative even numbers is 0. In other words, the question is whether we can choose the signs in these sequences in such a way that the sum becomes 0, both for odd and for even jumps.

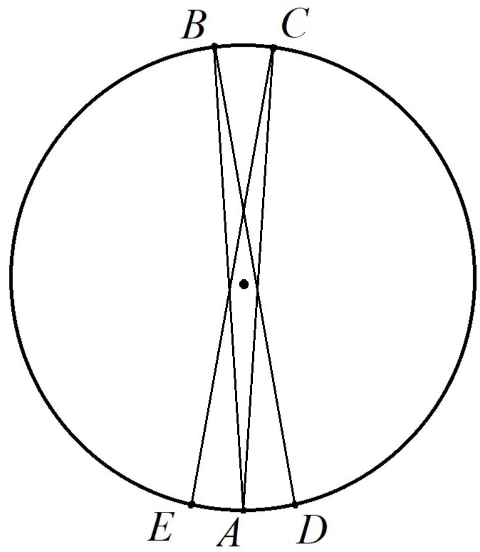
(a) In the case of 71 jumps, this is possible: for example, by using groups of four in a row (separately for odd and even), with signs $+ - - +$ within such a group giving 0 sum. Namely, for the 36 odd jumps we have 9 such groups: $(1-3-5+7)+(9-11-13+15)+\cdots+(65-67-69+71) = 0$. For the 35 even jumps, we have 8 groups of four from 4 to 66, and the remaining 3 numbers are $2+68-70 = 0$, so altogether: $2 + (4 - 6 - 8 + 10) + (12 - 14 - 16 + 18) + \cdots + (60 - 62 - 64 + 66) + 68 - 70 = 0$.

(b) In the case of 2022 jumps this is impossible: there are 1011 odd jumps, and any sum of positive or negative odd numbers is odd if the number of terms is odd, and therefore it cannot be equal to 0.

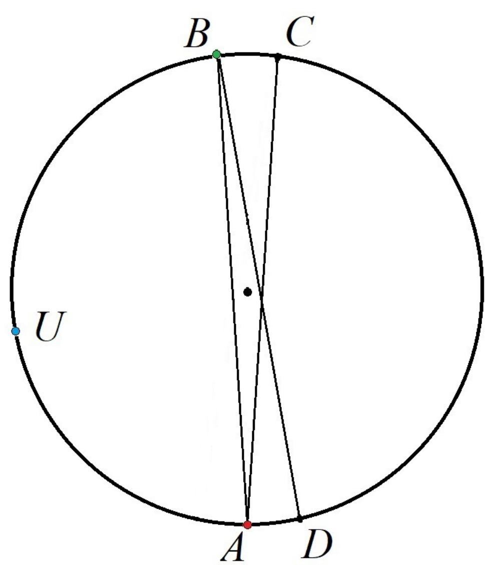
**Comments on submissions and solutions of Problem 4.** There are various other answers in part (a).

**Problem 5.** A regular 2021-gon is inscribed in a circle with centre $O$. Each vertex of the 2021-gon is coloured with one of three colours blue, green, or red, and each of these colours is used for at least one of the vertices. Prove that it is always possible to choose three vertices, one blue, one green, and one red, such that the triangle formed by connecting these three vertices contains the centre $O$.

**Solution of Problem 5.** Since the number of vertices is odd, the other end of the diameter through any vertex $A$ is between two vertices, say, $B$ and $C$ as on the picture (which is not to scale). Let us call such diagonals $AB$ and $AC$ quasidiameters. Note that for every vertex $X$ 'on the right' of the quasidiameter $AB$ (that is, on the bigger arc $AB$) the triangle $\triangle ABX$ contains the centre $O$.




First we claim that there is a quasidiameter with different colours at its endpoints. Indeed, otherwise $A$ would have the same colour as both $B$ and $C$, then the same colour would be at the endpoints of the quasidiameters $BD$ and $CE$, where $E$ and $D$ are nearest vertices to $A$, and so on, this process would result in all vertices having the same colour, contrary to the hypothesis.




We assume without loss of generality that the vertices $A$ and $B$ are red and green. If there is a blue vertex $X$ on the right of the quaisdiameter $AB$, then we are done, $\triangle ABX$ is a required triangle. Therefore we can assume that all vertices on the right of the quaisdiameter $AB$ are red or green. It follows that there must be a blue vertex $U$ on the left of the quasidiameter $AB$. In particular, the vertices $C$ and $D$ are red or green. If $C$ is green, then $\triangle UAC$ is as required. If $D$ is red, then $\triangle UBD$ is as required. Hence we can assume that $C$ is red and $D$ is green. But then $\triangle UCD$ is as required.

**Comments on submissions and solutions of Problem 5.** There were several other correct solutions proposed by the participants.

**Problem 6.** Positive integers 1, 2, 3, $\ldots, 2022$ are arbitrarily divided into three groups each containing 674 numbers. Prove that it is always possible to choose three numbers, one from each group, such that one of these numbers is the sum of the other two.

**Solution of Problem 6.** We denote these three groups by $A, B, C$ and assume without loss of generality that $A$ contains all the numbers $1, 2, \ldots, k-1$, while $k$ is in $B$ (here it is also possible that $k = 2$, when 1 is in $A$, and 2 in $B$). Say for brevity that a triple of numbers is good if these three number are from different groups and the sum of two of them is equal to the third. We argue by contradiction, that is, suppose that there are no good triples, and aim at something contradicting the hypothesis. Namely, assuming that there are no good triples, we claim that for any number $a$ in $C$ the number $a-1$ must be in $A$. This will imply that there are more numbers in $A$ than in $C$, contrary to the hypothesis.

Let $a$ be a number in $C$. Then $a-1$ is not in $B$, for otherwise $(1, a-1, a)$ would be a good triple. Suppose that $a-1$ is in $C$. Consider $a-k$. If $a-k$ is in $A$, then $(a-k, k, a)$ is a good triple, and if $a-k$ is in $B$, then $(k-1, a-k, a-1)$ is a good triple. Therefore $a-k$ must be in $C$. Similarly, if $a-k-1$ is in $A$, then $(a-k-1, k, a-1)$ is a good triple, and if $a-k-1$ is in $B$, then $(1, a-k-1, a-k)$ is a good triple. Therefore $a-k-1$ must also be in $C$. This process can be repeated with $a-k$ in place of $a$, then with $a-2k$, etc. As a result all numbers of the form $a-ik$ and $a-ik-1$ belong to $C$. (One can of course use here induction for a more rigorous argument.) But for some $i$ the number $a-ik$ will coincide with one of the numbers $1, 2, \ldots, k$ and therefore will be in $A$ or $B$. This contradiction proves that $a-1$ must be in $A$.

As a result, if $C$ consists of the numbers $c_1, \ldots, c_{674}$, the numbers $c_1 - 1, \ldots, c_{674} - 1$ all belong to $A$, and all these numbers are different and all greater than 1, since 2 is in $A$ or $B$. But $A$ also contains 1, so there will be more than 674 numbers in $A$, a contradiction.

**Comments on submissions and solutions of Problem 6.** There are other possible proofs by contradiction. Some submissions contained more complete arguments than others.