UNIVERSITY OF
LINCOLN

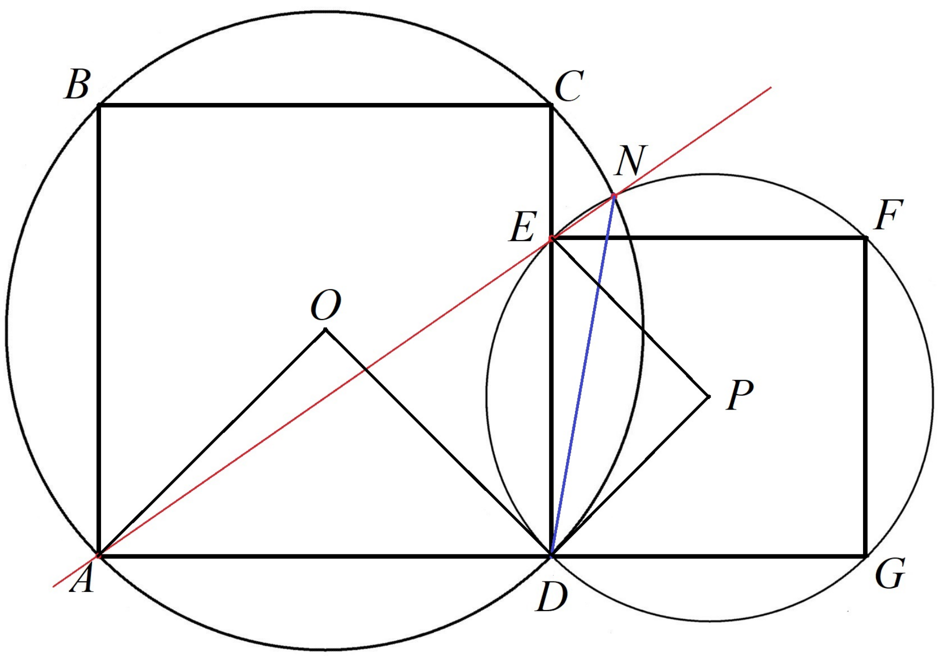
SCHOOL OF MATHEMATICS AND PHYSICS

Mathematics Challenge–2022–23

Brief solutions

Note that each problem may have several different solutions by various methods.

Problem 1. Two squares $ABCD$ and $DEFG$ are placed next to each other in such a way that D is a common vertex, while the vertex E of the second square is on the side CD of the first, as shown on the picture. Prove that the straight line through the points A and E passes through the intersection point N of the circles circumscribed about these squares.



Solution of Problem 1. We have $\angle AND = 45°$, since $\angle AOD = 90°$, where O is the centre of the circle circumscribed about the square $ABCD$. Similarly, $\angle END = 45°$, since $\angle EPD = 90°$, where P is the centre of the circle circumscribed about the square $DEFG$. Hence the points A, E, N are on the same straight line (which passes through N making angle 45° with ND).

Comments on submissions and solutions of Problem 1. There are various other solutions suggested by the contestants. For example, one can use coordinate geometry, finding the coordinates of N, etc.

Problem 2. Consider the following infinite triangular table:

				1				
			1	1	1			
		1	2	3	2	1		
	1	3	6	7	6	3	1	
1	4	10	16	19	16	10	4	1
.	.	.	.	.	.	.	.	.

The first row consists of a single 1. All subsequent rows are calculated as follows: every number is the sum of the three entries adjacent to it in the row above (where blank entries count as zero). For example, the three adjacent entries above 4 are 0, 1 and 3, and of course 4 = 0+1+3. Prove that every row, starting from the third one, contains an even number.

Solution of Problem 2. One solution is to notice that the four left-most (non-zero) entries of each row are determined by the four left-most entries in the row immediately above. Since we are only interested in the parity, we can write just "o" and "e" for odd and even and use the usual rules "o + o = e", "o + e = o", "e + e = e". We see that the pattern starts repeating itself from row 7: starting from row 3 and writing only the four left-most entries, we have

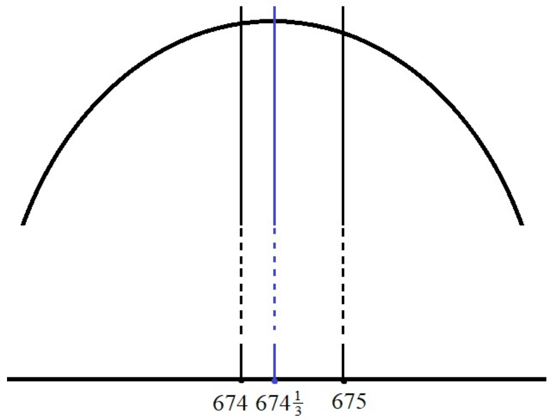
row 3: o e o e
row 4: o o e o
row 5: o e e e
row 6: o o o e
row 7: o e o e

and then the same pattern will be repeating: row 8 will be the same as row 4, since row 7 is the same as row 3, etc. We see that there "e" in each of these initial four entries, which proves the claim.

Comments on submissions and solutions of Problem 2. The contestants submitted several other valid proofs, with similar arguments, some using the fact that the middle number is always odd, etc.

Problem 3. If x, y, z are positive integers such that $x+y+z = 2023$, what is the maximum possible value of $xy + xz + yz$?

Solution of Problem 3. If we were looking for the maximum values over triples of any positive real numbers x, y, z such that $x + y + z = 2023$, then it would be attained for $x = y = z$, as can be shown (see comments below). But in this problem we must stick to positive integers, and therefore should be careful when proving that the maximum is attained at a particular triple. Here is one possible rigorous argument. Since the sum $x + y + z = 2023$ is odd, at least one of the numbers must be odd. Without loss of generality, let x be odd. When x is fixed, the sum $y + z$ is fixed, and then the value of $xy + xz + yz = x(y + z) + yz$ is maximal when yz is maximal. For a given sum $y + z$, which is even, the product yz is maximal for $y = z$, which are integers. Let $u = y = z$; then $xy + xz + yz = x(y + z) + yz = 2xu + u^2 = 2(2023 - 2u)u + u^2 = 4046u - 3u^2$. This quadratic function $f(u)$ of u attains maximal value at $u = 674\frac{1}{3}$ when u is regarded as a real variable. Since the graph of this quadratic parabola is symmetric with respect to the vertical line through $674\frac{1}{3}$ and the function increases for $u < 674\frac{1}{3}$ and decreases for $u > 674\frac{1}{3}$, the maximal value for integer u is attained at the closest integer to $674\frac{1}{3}$, that is, at $u = 674$; see the picture. Therefore the maximum of $xy + xz + yz$ is attained at $y = z = 674$, whence $x = 675$, and this maximum is 1,364,176.



(Not to scale)

Comments on submissions and solutions of Problem 3. As observed in some submissions, since $2023^2 = (x + y + z)^2 = (x^2 + y^2 + z^2) + 2(xy + xz + yz)$, the maximum value of $xy + xz + yz$ corresponds to the minimum value of $x^2 + y^2 + z^2$. Geometrically, all (real) points in 3D coordinate space with $x + y + z = 2023$ form the plane intercepting the axes at 2023, while $x^2 + y^2 + z^2$ is the square of the distance to the origin by Pythagoras. Thus, we need a point on this plane (in the positive octant) that is nearest to the origin. If we were looking for any real x, y, z, this point would be the base of the perpendicular dropped from the origin onto this plane. By the symmetry, this point has coordinates $x = y = z = 2023/3$. But we need a point with integer coordinates. To have minimum distance to the origin, by Pythagoras this integer point must be one of the closest to that base of perpendicular. Some work is still needed to accurately find this integer point (with a proof), and the result is of course the same as above. One of the contestants cleverly noticed that the values $x = y = z = 2023/3$ produce $xy + xz + yz = 1,364,176\frac{1}{3}$, which is the maximum over all real numbers as shown by this argument. Then for integers the maximum cannot be bigger than 1,364,176, and this value is indeed is attained at $y = z = 674$, $x = 675$.

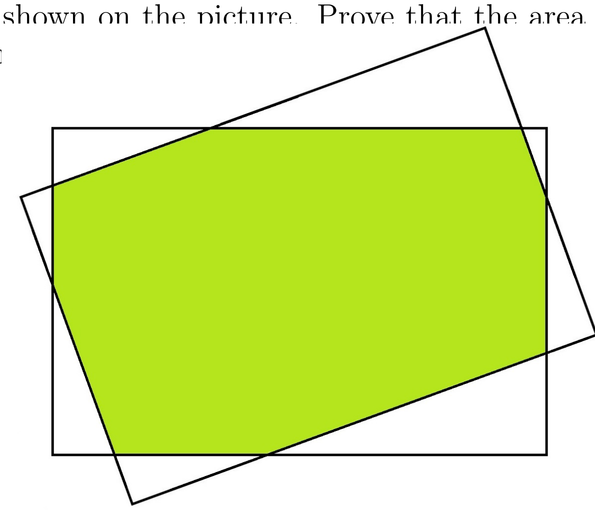
Full marks were given only for solutions that rigorously justified the results; it was not enough to simply write, say, that "obviously, the values of x, y, z must be as close to each other as possible", even if this 'argument' leads to the same final answer.

Problem 4. The integers $1, 2, 3, \ldots, 2022$ are rearranged in some order $a_1, a_2, \ldots, a_{2022}$ in such a way that the sum of any two consecutive numbers is at most 2512, that is, $a_i + a_{i+1} \leqslant 2512$ for every $i = 1, 2, \ldots, 2021$. Prove that there is j such that $a_j + a_{j+2} > 2512$.

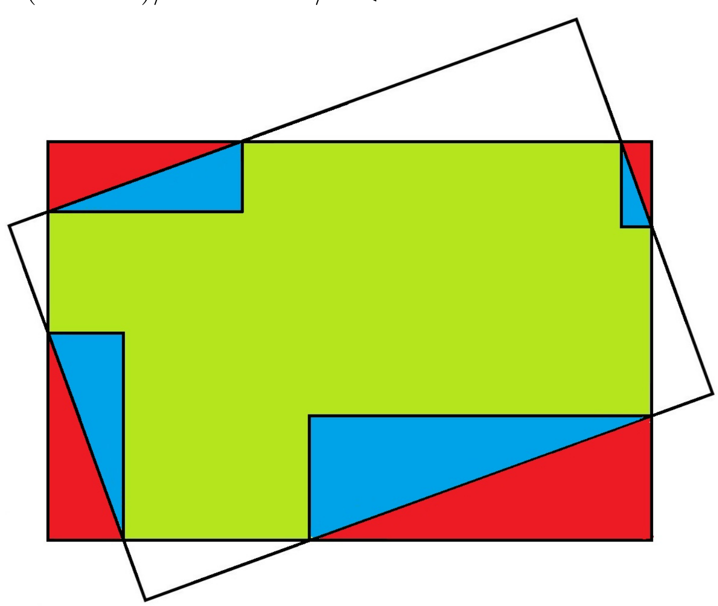
Solution of Problem 4. First we note that if a and b are any two different numbers among 1256, 1257, ..., 2022, then $a + b > 2512$, simply because this is true for he smallest possible pair out of this list. This list contains $2022 - 1255 = 767$ numbers. We divide the sequence $a_1, a_2, \ldots, a_{2022}$ into triples $a_1, a_2, a_3 \mid a_4, a_5, a_6 \mid \cdots \mid a_{2020}, a_{2021}, a_{2022}$. This is possible, since 2022 is divisible by 3, and we obtain $2022/3 = 674$ such triples. Since the list above contains 767 numbers, more than 674, there is a triple a_j, a_{j+1}, a_{j+2} containing at least two numbers from that list. These two cannot be a_j, a_{j+1} or a_{j+1}, a_{j+2} by the hypothesis. Hence, a_j and a_{j+2} are from the list, so $a_j + a_{j+2} > 2512$, as required.

Comments on submissions and solutions of Problem 4. In some submissions attempts were made by considering 'the worst possible' case, but this type of argument is difficult to fully and rigorously justify.

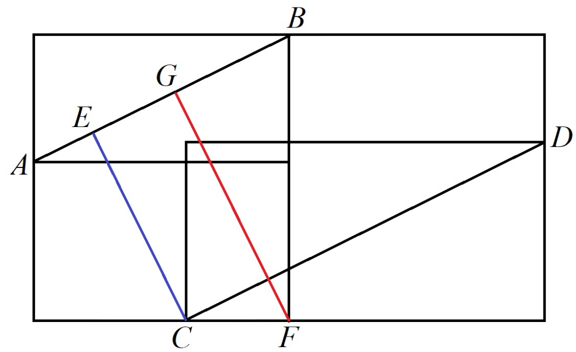
Problem 5. Two congruent rectangles are situated on the plane in such a way that their sides intersect in eight points as shown on the picture. Prove that the area of the common part of these rectangles is at least half th



Solution of Problem 5. One of the solutions is based on the following picture, where the red bits of one of the rectangles (the 'horizontal' one) that are outside of the intersection are reflected in the sides of the other rectangle as blue bits. If we manage to prove that the four blue bits are always disjoint, then we clearly see that the area of the red outside bits is equal to the area of the blue inside bits; but the intersection also has the (remaining) green area, so the intersection area is more than the outside area, and therefore more than half of the total area of the rectangle. The same argument can be made using formulas: with areas denoted by R, B, G, we have $R+B+G=A=$ the area of the rectangle, and $B+G=I=$ the area of the intersection; since $R=B$, we have $A/2=(R+B+G)/2=(2B+G)/2=B+G/2\leqslant B+G=I$.

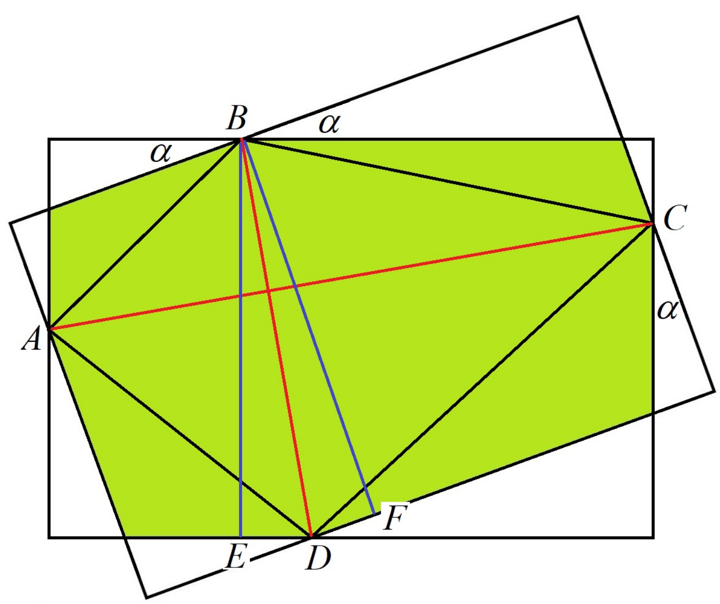


But this argument is complete only if we prove that the blue triangles are always disjoint. Any two blue triangles reflected in adjacent sides (that is, reflected from red ones at adjacent corners) are clearly disjoint, since they have parallel sides with some green area separating them. Thus it remains to prove that the opposite blue triangles are disjoint. Arguing by contradiction, suppose that two such blue triangles are intersecting, as on the following picture, where AB and CD are parts of sides of the second 'oblique' rectangle.



Let CE be the perpendicular dropped from C onto the opposite side of the second rectangle (it does not matter whether E is inside or outside of the first 'horizontal' rectangle), let BF be the perpendicular dropped from B onto the opposite side of the first rectangle, and let FG be the perpendicular dropped from F onto the opposite side of the second rectangle. Note that the point F is on the right of C because the blue triangles intersect by our assumption. Then the length of CE is equal to the length of the side of the rectangle, the same as BF. But $BF > FG > CE$, a contradiction. Therefore those opposite blue triangles cannot intersect. (Note, for the benefit of the second solution below, that we actually proved that the base F of the perpendicular BF cannot be on the right of the intersection point C of the long sides of the rectangles.)

Another solution is using the quadrangle $ABCD$ the vertices of which are intersections of long sides with long sides of the rectangles, and short with short sides.



One can prove that the diagonals AC and BD are perpendicular to each other, and then the area of $ABCD$ is equal to $AC \cdot BD/2$. Since the length AC is at least the side of the rectangle, and BD at least the other side, we obtain that the area of $ABCD$ is at least half the area of the rectangle, and $ABCD$ is only a part of the intersection. To prove that $AC \perp BD$, we find the angle that BD makes with the vertical side of the 'horizontal' rectangle, and then the same calculation shows that

AC makes the same angle with the horizontal side. Namely, this angle is $\alpha/2$, where α is the angle between the long sides of the rectangles. Indeed, drop the perpendiculars BE and BF onto the long sides of the rectangles as on the picture; note that $\angle EBF = \alpha$. Consider $\triangle BDE$ and $\triangle BDF$. These are right triangles with common hypothenuse and equal sides $BE = BF$. Hence they are congruent, so $\angle DBE = \angle DBF = \angle EBF/2 = \alpha/2$. This argument implicitly uses the fact that D is on the right of E, but this is always the case by the same argument as in the first solution above.

Comments on submissions and solutions of Problem 5. Some submissions contained arguments similar to the first solution above, when parts of a rectangle outside the intersection are 'reflected into the intersection'. But not all such attempts included explanation that the resulting areas in the intersection are disjoint, without which the proof is incomplete.

Problem 6. Prove that for any sequence of 2023 positive real numbers $r_1, r_2, \ldots, r_{2023}$ one can find a positive integer $k \leqslant 2023$ such that each of the k numbers

$$r_k, \quad \frac{r_k + r_{k-1}}{2}, \quad \frac{r_k + r_{k-1} + r_{k-2}}{3}, \quad \ldots \quad, \frac{r_k + r_{k-1} + \cdots + r_1}{k}$$

is not greater than $\dfrac{r_1 + r_2 + \cdots + r_{2023}}{2023}$.

(It is also possible that $k = 1$, when those k numbers consist of only r_k, or $k = 2$, when those k numbers consist of only r_k and $(r_k + r_{k-1})/2$.)

Solution of Problem 6. Let for brevity $M = \dfrac{r_1 + r_2 + \cdots + r_{2023}}{2023}$, which is the mean (average) of all numbers. Let m be the minimum (first) number such that

$$\frac{r_m + r_{m-1} + \cdots + r_1}{m} \leqslant M.$$

Such a number m exists since we of course have

$$\frac{r_{2023} + r_{2023-1} + \cdots + r_1}{2023} = M.$$

Then we can put $k = m$, and the required conditions will be satisfied. Indeed, if for some j the mean of $r_{j+1}, \ldots, r_m$ is greater than M, then the mean of $r_1, \ldots, r_j$ must be less than M, in order for the mean of all $r_1, \ldots, r_j, r_{j+1}, \ldots, r_m$ to be at most M. This, however, contradicts the choice of m as the first index with this property. The same argument can be written using inequalities, of course.